Working Through Bell's 1964 Paper

1. Introduction.

John Bell was a personal hero to me. His comments on the danger that his inequalities presented to special relativity resonated with my thinking. I found his comments to be insightful regarding the problem of understanding where and how quantum collapse occurs. He was, like I am, supportive of first teaching students Lorentz's approach to the Lorentz transformation equations prior to introducing the ideas of relativity.

Prior to the internet, I exchanged letters with John Bell concerning an idea I had that the length contraction may not physically exist. His comments during that exchange led me to realize an error in my original thinking that led to my later work on that subject. I was grateful that such a well-known senior person would take the time to meaningfully respond to me.

While I earlier accepted the main results of Bell's 1964 paper, I had not worked through the intermediate steps involved in deriving them. Those intermediate steps now appear in the derivations below. Note that when working through this paper, it will be important to have a copy of Bell's 1964 paper to refer to, as the entire point of this paper is to work through that 1964 paper by providing intermediate steps and explanations that aren't present in that classic work. For that reason, the 1964 paper is included as an appendix.

Before we begin, some stylistic notes: I am adopting equation numbering wherein equations taken from Bell's 1964 paper will begin with B. In some cases, Bell presents un-numbered equations; in those cases, equations will begin with BX. My own equations will not have a preceding B nor a BX. Also, here I'll add statements in red for things I believe to be erroneous (typos) in Bell's paper; in green for evaluations of equality sub-conditions of Bell's inequalities, and in purple for commentary I wish to emphasize. The red, green, and purple statements will have heading numbers starting with R, G and P, respectively.

2. Bell's First Illustration in his Section III.

For me, the first problem in understanding Bell's 1964 paper is Eq. (B2). The integral is defined without specifying what the limits of integration are, and the normalization is merely assumed. By itself, that is a common practice, but when trying to understand Bell's theorem it can be helpful to provide these factors. Here is Eq. (B2):

$$P(\mathbf{a}, \mathbf{b}) = \int d\lambda\, \rho(\lambda)\, A(\mathbf{a}, \lambda)\, B(\mathbf{b}, \lambda) \qquad \text{(B2)}$$

In Eq. (B2), $P(\mathbf{a}, \mathbf{b})$ is the expectation value of A times B, $\rho(\lambda)$ is the density function of the hidden variable λ, and the expectation value is obtained by integrating over all values of the hidden variable. $A(\mathbf{a}, \lambda)$ is the measurement of one quantum by a detector oriented in such a way as to find the spin component in direction of **a**, and B(**b**, λ) is the measurement of a partner quantum by a distant second detector oriented to find the spin component in the direction of **b**. The two quanta are assumed to be formed in an entangled state such that their spins are in opposing directions. They then freely propagate away from each other to be measured at detectors A and B, which can be an arbitrary distance apart. We'll provide further analysis below.

Bell gives three illustrations in Section III of his paper. In the first illustration it is assumed that the initial spin of the quantum is oriented in the **p** direction, and it then encounters a detector oriented to find the spin component in the direction **a**. Bell stipulates an example distribution of the hidden variable λ such that that λ is a unit vector with uniform probability distribution over a hemisphere. Bell then asserts that the result of a measurement of the spin component of the quantum measured by a detector oriented to find the spin component in the direction of **a** is

sign $\lambda \cdot \mathbf{a}'$, (B4)

where Bell stipulates that **a'** will be specified later.

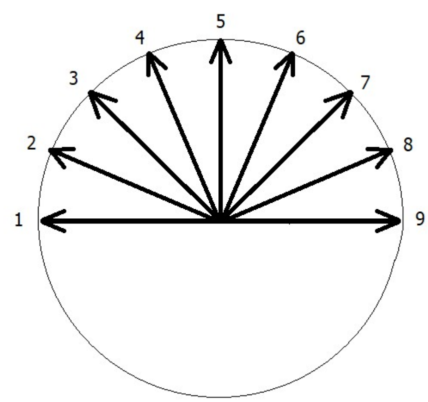

Figure 1. Uniform distribution of Hidden Variable Spins over a Hemisphere. Figure shows Equatorial Plane of an Associated Sphere.

The hidden variable distribution described by Bell is visualized in Figure 1, where several individual unit vectors are shown which begin in the center of a circle and end at the upper half of that circle. The circle shown is that of the equatorial plane of a sphere resulting from the tips of such vectors. In the figure, a rotation from one vector to the next will define the azimuthal angle. The polar angle is with respect to the axis perpendicular to the plane of the figure.

If **a'** is aligned with **p** (**p** is defined as case 5 in the figure) we can easily see that every λ between cases 1 and 9 will have an **a'** component such that A(**p**, λ) = 1. But now consider the case where **a'** is instead aligned with case 6 in the figure, which is a 22.5 degree azimuthal offset from **p**. In this case, the portion of the diagram between vectors 1 and 2 will now have sign $\lambda \cdot \mathbf{a}' = -1$ instead of +1 and so it will reduce the expectation value by $2\theta'/\pi$, where θ' is the angle between **a'** and **p**. This is because we no longer have the +1 contribution from that region, and furthermore we now have a –1 contribution from that region. This is what gives us Bell's Eq. (B5),

$$<\sigma \cdot \mathbf{a}> = 1 - 2\theta'/\pi \qquad \text{(B5)}$$

While easy to see how Eq. (B5) arises from Figure 1 and the above paragraph, it is also helpful to put things into an integral formulation beginning with an equation similar to Bell's Eq. (B2) so that we can obtain bounds and normalization of the integral. We form:

$$P(\mathbf{a}) = \int d\lambda\, \rho(\lambda)\, A(\mathbf{a}, \lambda) \qquad (1)$$

When **a** = **p** we know the value of the integral is 1, and below we apply normalization and bounds to enable this:

$$P(\mathbf{p}) = \int d\lambda\, \rho(\lambda)\, A(\mathbf{p}, \lambda) = (1/\pi)\,{}_{-\pi/2}\!\int^{\pi/2} d\varphi = 1 \qquad (2)$$

In Eq. (2) we have defined an azimuthal angle φ between **p** (or our unit vector 5) and λ, and the zenith direction is defined as being perpendicular to both **a** and **p** for our analysis. Integrating over λ means integrating over the polar angle ψ as well as φ since each unit vector λ is associated with specific values of ψ and φ. The integration over the polar angle is independent of φ. With ψ the polar angle, we get ${}_0\!\int^{\pi} \sin\psi\, d\psi = 2$, and a normalization factor of 1/2 leaves us with Eq. (2). We then integrate φ from $-\pi/2 <= \psi <= \pi/2$ to cover the range of λ values (a hemisphere) for Bell's first illustration. (Bell's first illustration is that which includes Eqs. (B4) through (B7).) In Eq. (2) we have A(**p**, λ) = 1 and $\rho(\lambda) = 1$ over the hemisphere domain of integration, and we use a normalization ($1/\pi$) so that the integral is unity.

So far so good, but now it is where things get tricky as we must confront the meaning of **a'**. Bell states:

(Begin quote from Bell's paper.)

Suppose then that **a'** is obtained from **a** by rotation towards **p** until

$$1 - 2\theta'/\pi = \cos\theta \qquad \text{(B6)}$$

where θ is the angle between **a** and **p**. Then we have the desired result

$$<\sigma \cdot \mathbf{a}> = \cos\theta \qquad \text{(B7)}$$

(End quote from Bell's paper.)

What is going on here? When the detector is oriented along **a**, if we assume that the hidden variables exist in the manner discussed above and shown in Figure 1, we must theorize that the underlying physics requires the **a'** given by Bell through Eq. (B6) in order to match the empirical results. That is, while it may seem obvious that the usual quantum mechanics prediction of a value of $\cos\theta$ is the underlying physics, as directly obtained from **a** and **p** ($\cos\theta = \mathbf{a} \cdot \mathbf{p}/ap$), this is not the case now. Instead, the underlying physics involves θ' as used in Eq. (B5), $\langle\sigma \cdot \mathbf{a}\rangle = 1 - 2\theta'/\pi$, where θ' is the angle between **a'** and **p**, and an **a'** obtained from **a** and **p** by rotation until Eq. (B6) is obtained. The hidden variable proposal forces us to accept this relation to meet the empirical result, but once done, as Bell says, each individual measurement can result from a specific value of λ and the statistical nature arises because we do not know what that value is. The expectation value comes out as well.

But why isn't **a'** = **a**? Well, for some particular values, it is. When **a** is parallel to **p**, $\theta = 0$, and $\cos\theta = 1$. Eq. (B6), $1 - 2\theta'/\pi = \cos\theta$, is satisfied in this case if $\theta' = 0$, which means **a'** is parallel to **p**, and we have $\theta' = \theta$, which is consistent with **a'** = **a**. And when **a** is perpendicular to **p**, $\theta = \pi/2$, and $\cos\theta = 0$. Eq. (B6), $1 - 2\theta'/\pi = \cos\theta$, is satisfied in this case if $\theta' = \pi/2$, which means **a'** is perpendicular to **p**, and we have $\theta' = \theta$, which is consistent with **a'** = **a**. Yet now consider $\theta = \pi/4$. In this case, if we still try to keep **a'** = **a** and $\theta' = \theta$, Eq. (B6), $1 - 2\theta'/\pi = \cos\theta$ leaves us with $1 - 2\theta'/\pi = 1 - 2(\pi/4)/\pi = 0.5 \neq \cos(\pi/4) = \mathrm{sqrt}(2)/2$ and hence Eq. (B6) is not satisfied when $\theta = \pi/4 = 0.785$ radians. Since θ is the angle of the detector with respect to **p**, the only thing we can do in this situation is to adjust θ' such that $1 - 2\theta'/\pi = \mathrm{sqrt}(2)/2$. This is done by setting $\theta' = \pi[1 - \mathrm{sqrt}(2)/2]/2 = 0.460$ radians, which is a rotation of **a'** from its starting point of **a** (with $\theta = 0.785$ radians) toward **p** (at $\varphi = 0$ radians), and this is what Bell says is needed here.

As a final note before leaving Bell's First Illustration, note that θ' cannot be negative. If it were, the expectation value given in Eq. (B5), $\langle\sigma \cdot \mathbf{a}\rangle = 1 - 2\theta'/\pi$, would be greater than one. Nor can θ' exceed π, because then the expectation value given in Eq. (B5) will be less than minus one. And yet **a** must be allowed to have a domain from $-\pi < \theta < \pi$, and so what this tells us is that Bell is using the absolute values of the domains θ and θ' in his paper.

To reinforce the main point, introduction of the hidden variable that is represented in Figure 1 results in a requirement that the underlying physics involves Eq. (B5) above, as well as Eq. (B6) to set **a'**. Once that is done, a hidden variable theory is possible for single spin measurements of a pure state polarized along **p**, although it does introduce some complexity as compared to the simpler quantum mechanical relation $\langle\sigma \cdot \mathbf{a}\rangle = \cos\theta$.

3. Bell's Second Illustration in his Section III.

Bell states in his second illustration that λ is a unit vector uniformly distributed over all directions, which is here depicted in Figure 2. When the original spin zero state decays it is reasonable to assume that it will do so in an isotropic way, and this can be understood as leading to Bell's hidden variable proposal.

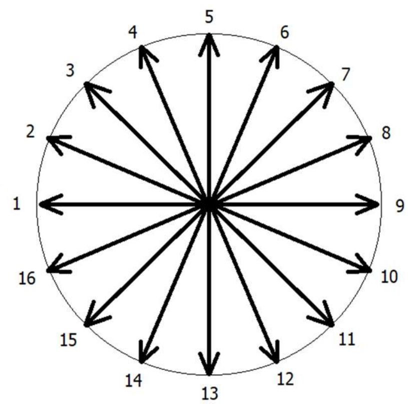

Figure 2. Uniform Distribution of Hidden Variable Spins over a Sphere. Figure shows Equatorial Plane of the Sphere.

We'll now start on the derivation of Eq. (B10) and we see that the second of Bell's Eqs. (B9), B(a, b) = – sign(**b** · λ), is a typo, as it should be

B(**b**, λ) = – sign(**b** · λ) (3)

R1. Typo One – Bell's erroneous second of Eqs. (B9), which should instead be our Eq. (3).

The first of Bell's Eqs. (B9) is fine, A(**a**, λ) = sign(**a** · λ). These equations result from the physics. If a central system of total spin 0 decays into particles moving back-to-back, then the particle heading to A will have one spin and the particle heading to B will have the opposite spin. We can choose the one going to A as having spin λ, and that means the one going to B has spin –λ, and this then results in Eqs. (B9) once we correct the typo via our Eq. (3).

Similar to our analysis above in our section 2 we will form λ as a hidden variable unit vector, that this time again is described by the angles φ and ψ, although this time the unit vectors have a constant density over the entire sphere. Various instances of λ are shown in Figure 2.

Without loss of generality, we can assume **a** is aligned with unit vector 5 as shown in Figure 2, and for our first case of analysis we will assume **b** is aligned with **a**. In this case, for the top half of Figure 2, A(**a**, λ) = +1 and B(**b**, λ) = B(**a**, λ) = –1. For the bottom half of Figure 2, A(**a**, λ) = –1 and B(**b**, λ) = B(**a**, λ) = +1. Hence everywhere A(**a**, λ)B(**a**, λ) = –1. We then use Eq. (B2) to arrive at:

$$P(\mathbf{a}, \mathbf{b}) = P(\mathbf{a}, \mathbf{a}) = \int d\lambda\, \rho(\lambda)\, A(\mathbf{a}, \lambda)\, B(\mathbf{a}, \lambda) = (1/2\pi) \int_{-\pi}^{\pi} -d\varphi = -1 \qquad (\mathbf{a} \text{ parallel to } \mathbf{b}) \qquad (4)$$

In Eq. (4) we have defined an azimuthal angle φ between **a** and λ, and the zenith direction is defined as being perpendicular to both **a** and **b** for our analysis. The integration over the polar angle is independent of φ. With ψ the polar angle, we get $\int_0^{\pi} \sin\psi \, d\psi = 2$, and a normalization factor of 1/2 leaves us with Eq. (4). Integrating over λ again means integrating over both the azimuthal angle φ, and the polar angle ψ since each unit vector λ is associated with specific values of φ and ψ. This time the domain of ψ is again $0 <= \psi <= \pi$ and the domain of φ is now $-\pi <= \varphi <= \pi$ to cover the range of λ values (the full sphere) for Bell's second illustration. (Bell's second illustration is the one that includes Eqs. (B8) through (B10).) In Eq. (4) we have A(**a**, λ)B(**a**, λ) = –1 and $\rho(\lambda)$ = 1 over the sphere domain of integration, and we use a normalization ($1/2\pi$) so that the integral is –1.

Next, we consider the case when **a** is aligned along unit vector 5 in Figure 2, while **b** is aligned with unit vector 6. In this case, when λ is aligned with **a** we have A(**a**, λ) = +1 and B(**b**, λ) = –1 and these values of A and B are retained up to the point where λ is aligned with unit vector 9 in Figure 2. But when λ is between unit vectors 9 and 10 we have A(**a**, λ) = –1 and B(**b**, λ) = –1. And when λ is between unit vectors 10 and 13 we have A(**a**, λ) = –1 and B(**b**, λ) = +1. We see that the portion of the integration when λ is between unit vectors 9 and 10 increases the value of the integral by twice the angle between **a** and **b**; one factor of the increase comes from no longer having the negative contribution and the second factor of the increase comes from what is now a positive contribution. Continuing the integration around the full circle results in an increase of the value of the integral by four times the angle between **a** and **b**, due to the effects from the region between unit vectors 9 and 10 and also from the region between unit vectors 1 and 2. Defining the angle between **a** and **b** to be θ, and realizing that we must include the normalization factor of $1/2\pi$ as we did for Eq. (4) leaves Bell's Eq. (B10):

$$P(\mathbf{a}, \mathbf{b}) = \int d\lambda\, \rho(\lambda)\, A(\mathbf{a}, \lambda)\, B(\mathbf{b}, \lambda) = (1/2\pi) {}_{-\pi}\!\int^{\pi} -d\varphi = -1 + 2\theta/\pi \qquad \text{(general case)} \qquad \text{(B10)}$$

Notice here that θ can only be defined over the region $0 <= \theta <= \pi$. Like the first illustration, Bell uses the absolute values of θ.

It is readily apparent that Eq. (B10) results in the equations (B8):

$$P(\mathbf{a}, \mathbf{a}) = -P(\mathbf{a}, -\mathbf{a}) = -1\ ;$$

$$P(\mathbf{a}, \mathbf{b}) = 0 \text{ if } \mathbf{a} \cdot \mathbf{b} = 0 \qquad \text{(B8)}$$

(For P(**a, a**), θ is 0; for P(**a, –a**) θ is π; and when $\mathbf{a} \cdot \mathbf{b} = 0$, θ is $\pi/2$.)

Here, note that maximum discrepancy between the quantum prediction and that of the uniform hidden variable model can be found by differentiation. The discrepancy will be $D = \cos\theta - (1 - 2\theta/\pi)$. Taking the derivative with respect to θ, $dD/d\theta = -\sin\theta + 2/\pi$. D will have an extremum when $dD/d\theta = 0$, or, when $\sin\theta = 2/\pi$, or at $\theta = \sin^{-1}(2/\pi) = 0.6901$ radians = 39.54 degrees. The second derivative of D is $d^2D/d\theta^2 = -\cos\theta$, which is negative for $\theta = 0.6901$ radians, indicating that $\theta = 0.6901$ radians results in the maximum discrepancy. This is not too far from an example later used by Bell of $\theta = \pi/4$ radians = 0.785 radians = 45 degrees which results in $D = \cos\theta - (1 - 2\theta/\pi) = \mathrm{sqrt}(2)/2 - 1/2$.

4. Bell's Third Illustration in his Section III.

Bell's third illustration in his Section III is that we can achieve the quantum mechanical relation $<\sigma_1 \cdot \mathbf{a}\ \sigma_2 \cdot \mathbf{b}> = -\mathbf{a} \cdot \mathbf{b} = -\cos\theta$ if we replace **a** in Eq. (B9) by **a'** and allow **a'** to rotate from **a** toward **b** such that $1 - 2\theta'/\pi = \cos\theta$. Since θ' is defined with respect to both **a** and **b**, it is necessary for us to involve both **a** and **b** if we are to obtain the quantum mechanical relation. As Bell states, this is what we wish to avoid, as we do not want a result at A being influenced by what happens at B.

5. The Importance of the Illustrations.

The illustrations may already be enough to rule out the possibility of hidden variables, at least the plausible ones. What other assumption of a hidden variable would there be, other than one that is uniformly distributed over a sphere? Sure, inventive minds might manufacture some hideous concoction to fit results, but what we are looking at are paired spin states leaving from a central spin zero decay. The utter simplicity of what we are investigating should have a very simple underpinning, and it should be isotropic since there is no physical reason for it not to be. Nonetheless, Bell goes on in his next section to investigate a more general proof that hidden variables will lead to predictions different from quantum mechanics.

6. Bell's First Contradiction in his Section IV.

Bell's section IV contains two distinct portions. The first portion ends with the paragraph following Eq. (B15), and we'll consider that portion here in our section 6.

We'll work through Bell's section IV by commenting on each equation. Eq. (B12), $\int d\lambda\, \rho(\lambda) = 1$, just sets the normalization of the hidden variable density $\rho(\lambda)$. Eq. (B13), $A(\mathbf{a}, \lambda) = -B(\mathbf{a}, \lambda)$ is simply a statement that, for a given hidden variable λ, the result at detector A must be the opposite of that at B if the detectors are oriented in the same direction, since the spin of the particle interacting with A is the opposite of the spin of the particle interacting with B. To get to Eq. (B14), we start with Eq. (B2):

$$P(\mathbf{a}, \mathbf{b}) = \int d\lambda\, \rho(\lambda)\, A(\mathbf{a}, \lambda)\, B(\mathbf{b}, \lambda) \qquad \text{(B2)}$$

and substitute in Eq. (B13), now rewritten as $B(\mathbf{b}, \lambda) = -A(\mathbf{b}, \lambda)$, to get:

$$P(\mathbf{a}, \mathbf{b}) = -\int d\lambda\, \rho(\lambda)\, A(\mathbf{a}, \lambda)\, A(\mathbf{b}, \lambda) \qquad \text{(B14)}$$

Bell then has a couple of un-numbered equations, the first is

$P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c}) = -\int d\lambda\, \rho(\lambda)\, [A(\mathbf{a}, \lambda)\, A(\mathbf{b}, \lambda) - A(\mathbf{a}, \lambda)\, A(\mathbf{c}, \lambda)]$

$= \int d\lambda\, \rho(\lambda)\, A(\mathbf{a}, \lambda)\, A(\mathbf{b}, \lambda)\, [A(\mathbf{b}, \lambda)\, A(\mathbf{c}, \lambda) - 1]$ (BX1)

This results since $A(\mathbf{b}, \lambda)\, A(\mathbf{b}, \lambda)$ is always 1, whether $A(\mathbf{b}, \lambda)$ is +1 or –1. And now, since $A(\mathbf{a}, \lambda)$ and $A(\mathbf{b}, \lambda)$ are always either +1 or –1,

$A(\mathbf{a}, \lambda)\, A(\mathbf{b}, \lambda) <= 1$ (5)

And Eq. (5) applied to Eq. (BX1) leads to

$|P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c})| <= \int d\lambda\, \rho(\lambda)\, [1 - A(\mathbf{b}, \lambda)\, A(\mathbf{c}, \lambda)]$ (BX2)

Eq. (BX2) is derived by first multiplying both sides of Eq. (BX1) by –1. Next, we take the absolute value of the left-hand side, noting that the right-hand side is a positive quantity. Then we note that over the integral, $A(\mathbf{a}, \lambda)\, A(\mathbf{b}, \lambda)$ will sometimes be 1, and unless $\mathbf{a} = \mathbf{b}$ it will sometimes be –1, and hence the inequality results.

G.1. The equality in Eq. (BX2) results when $\mathbf{a} = \mathbf{b}$ and when $\mathbf{a} = -\mathbf{b}$.

From Eq. (BX2) Bell obtains Eq. (B15)

$1 + P(\mathbf{b}, \mathbf{c}) >= |P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c})|$ (B15)

Eq. (B15) follows from Eq. (BX2) as $\int d\lambda\, \rho(\lambda) = 1$ from Eq. (B12), and from Eq. (B14) $P(\mathbf{b}, \mathbf{c}) = -\int d\lambda\, \rho(\lambda)\, A(\mathbf{b}, \lambda)\, A(\mathbf{c}, \lambda)$

G.2. The equality in Eq. (B15) results when $\mathbf{a} = \mathbf{b}$ and when $\mathbf{a} = -\mathbf{b}$ as it follows Eq. (BX2). Note that G.1 and G.2 play no role in obtaining any other inequality further down in the paper.

Next, Bell makes the assertion that 'Unless P is constant, the right hand side is in general of order $|\mathbf{b} - \mathbf{c}|$ for small $|\mathbf{b} - \mathbf{c}|$.' Let's examine this assertion.

For the uniform λ case discussed in section 3 above, we found Bell's Eq. (B10) $P(\mathbf{a}, \mathbf{b}) = -1 + 2\theta/\pi$ where θ is the angle between **a** and **b**, and we also get $P(\mathbf{a}, \mathbf{c}) = -1 + 2\theta''/\pi$ where θ'' is the angle between **a** and **c**. From this, $P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c}) = 2\theta/\pi - 2\theta''/\pi$, which is indeed of general order $|\mathbf{b} - \mathbf{c}|$ for small $|\mathbf{b} - \mathbf{c}|$. This may be true for many λ cases, but I don't see how it is necessarily universally true. Here, Bell uses the term "stationary" to apparently mean a vanishing first order term in the Taylor expansion. $\cos\theta$ does have a vanishing first derivative at $\theta=0$, and hence if $P(\mathbf{a}, \mathbf{b}) - P(\mathbf{a}, \mathbf{c})$ has a first order term, then we have a contradiction at small θ. But at this point Bell's assertion that 'Unless P is constant, the right hand side is in general of order |**b** – **c**| for small |**b** – **c** |' is not shown to be proven for all hidden variable possibilities.

P.1. Bell's assertion that 'Unless P is constant, the right hand side is in general of order |**b** – **c**| for small |**b** – **c** |' might be correct, but no proof is given by Bell in his paper. It is just an asserted statement. Since Bell turns out to be correct in all other important aspects of the paper, I'm guessing he had a proof somewhere, but the paper itself is too sparse in details for me to determine the veracity of this particular claim.

7. Bell's Second Contradiction in his Section IV.

We'll now move on to the second contradiction of Bell's section IV, which is a more general proof. Here Bell compares the expectation value of a hidden variable theory, $\underline{P}(\mathbf{a}, \mathbf{b})$, to the expectation value of quantum mechanics, $\underline{-\mathbf{a} \cdot \mathbf{b}}$. Here I am using an underline instead of Bell's bar above the quantities. (The underline means we are averaging over vectors **a'** and **b'** that lie within small angles of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$, respectively.) Bell then proposes that the absolute value of the difference is bounded by ε in his Eq. (B16)

$$|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| <= \varepsilon \qquad \text{(B16)}$$

However, as described below, Bell has a typo in Eq. (B16) as it should have been:

$$|\underline{P}(\mathbf{a}, \mathbf{b}) + \underline{\mathbf{a} \cdot \mathbf{b}}| <= \varepsilon \qquad (6)$$

R.2. Typo Two – Bell's Eq. (B16) should be replaced by our Eq. (6).

Note next that if the left-hand side of Eq. (6) is zero, the hidden-variable expectation value will agree with that of quantum mechanics and that zero will obey the inequality. So why does a proof showing a contradiction not use >= instead of the <= used in Eq. (B16)? The reason is that sometimes the left-hand side is zero (the case of no contradiction) as we've seen in Eqs. (B8) above. Yet the implicit assertion is that sometimes the left-hand side is not zero. Hence Eq. (B16) presents a bound on the range of what the left-hand side obtains. The crucial point then is to understand the equals portion of the less than or equals condition, and to show that ε for that equals portion cannot be arbitrarily small. This means that as we work through the proof we'll need to pay special attention to the equals portion of the inequality relationship. Where there are different equals conditions for different cases, our goal is to find the smallest possible value of ε.

G.3. For Eq. (6), the equals condition is obtained when ε equals the asserted maximum value obtainable by $|\underline{P}(\mathbf{a}, \mathbf{b}) + \underline{\mathbf{a} \cdot \mathbf{b}}|$. At this point we merely have an assertion that such a maximum value exists.

Next, let us look at Eq. (B17):

$|\underline{\mathbf{a}\cdot\mathbf{b}} - \mathbf{a}\cdot\mathbf{b}| <= \delta$ (B17)

In Eq. (B17) Bell defines the small range δ about which $\underline{\mathbf{a}\cdot\mathbf{b}}$ is allowed to have its average value vary. When **a'** and **b'** individually vary, $|\underline{\mathbf{a}\cdot\mathbf{b}} - \mathbf{a}\cdot\mathbf{b}|$ always falls within δ. Indeed, δ is the maximum value that $|\underline{\mathbf{a}\cdot\mathbf{b}} - \mathbf{a}\cdot\mathbf{b}|$ obtains as we vary **a'** and **b'** within the small limit we consider.

G.4. For Eq. (B17) the equals condition is what we allow δ to be when we vary **a** and **b**. This is again just an assertion of the method Bell will use.

We now move on to Bell's Eq. (B18):

$|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a}\cdot\mathbf{b}| <= \varepsilon + \delta$ (B18)

G.5. For Eq. (B18) the equals condition is obtained for the maximum possible ε plus whatever value we allow δ to be. (B18) continues development of the proposed bounding difference between the quantum and hidden-variable expectation values.

Eq. (B18) is obtained from Eqs. (6) and (B17). (B17) tells us that the most $\mathbf{a}\cdot\mathbf{b}$ can exceed $\underline{\mathbf{a}\cdot\mathbf{b}}$ by is δ, and hence $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a}\cdot\mathbf{b}|$ can be at most the value of $|\underline{P}(\mathbf{a}, \mathbf{b}) + \underline{\mathbf{a}\cdot\mathbf{b}}|$, which is ε as given in Eq. (6), plus δ, giving us (B18). And let us now inquire about the equality part of this. We've seen above in section 3 under Eq. (B10) that θ can only be defined over the region $0 < \theta < \pi$ where θ is the angle between **a** and **b**, and this allows $\mathbf{a}\cdot\mathbf{b}$ to vary from +1 to –1. (**a** and **b** are unit vectors.) Now **a'** and **b'** can vary in any direction away from **a** and **b**, respectively, and therefore the maximum value of δ can be obtained for some choice of **a'** and **b'**. Now **a'** and **b'** occupy a range, and Bell instructs us to average over that range. $\underline{P}(\mathbf{a}, \mathbf{b})$ is what is obtained when **a'** and **b'** are averaged over. We can now do the variation in any way we wish and for one condition of a specific **a'** and **b'** the equality holds.

P.2. It is unclear at this point is why averaging of variations limited by δ is needed at all.

Note that it is in Eq. (B18) that we see the evidence for the typo of Eq. (B16), because the left-hand side of Eq. (B16) is identical to the left-hand side of Eq. (B18) and hence Eq. (6) should replace Eq. (B16) as we have done here.

Eqs. (B19) and (B20) are clear, as they merely follow from earlier definitions.

$\underline{P}(\mathbf{a}, \mathbf{b}) = \int d\lambda\, \rho(\lambda)\, \underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{b}, \lambda)$ (B19)

$|\underline{A}(\mathbf{a}, \lambda)| <= 1$ and $|\underline{B}(\mathbf{b}, \lambda)| <= 1$ (B20)

Eq. (B21) has a typo (typo three) as it should have an integral sign in front of it:

$$\int d\lambda\, \rho(\lambda)\, [\underline{A}(\mathbf{b}, \lambda)\, \underline{B}(\mathbf{b}, \lambda) + 1] <= \varepsilon + \delta \qquad (7)$$

R.3. Typo Three – Eq. (B21) should include an integral sign like we do in Eq. (7).

As stated by Bell, Eq. (B21), now re-expressed via our Eq. (7), comes about by starting with Eq. (B19), $\underline{P}(\mathbf{a}, \mathbf{b}) = \int d\lambda\, \rho(\lambda)\, \underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{b}, \lambda)$. Recall that Eq. (B18) is $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| < = \varepsilon + \delta$ and setting $\mathbf{a} = \mathbf{b}$ this becomes $|\underline{P}(\mathbf{b}, \mathbf{b}) + 1| < = \varepsilon + \delta$ and then substituting in Eq. (B19) for the case where $\mathbf{a} = \mathbf{b}$ we get Eq. (7). (Here we use $|\underline{P}(\mathbf{a}, \mathbf{b})| <=1$ so $\underline{P}(\mathbf{a}, \mathbf{b}) + 1 >= 0$ and $|\underline{P}(\mathbf{a}, \mathbf{b}) + 1| = \underline{P}(\mathbf{a}, \mathbf{b}) + 1 < = \varepsilon + \delta$. With $\mathbf{a} = \mathbf{b}$, $\underline{P}(\mathbf{b}, \mathbf{b}) + 1 < = \varepsilon + \delta$. We also use the normalization $\int d\lambda\, \rho(\lambda) = 1$.)

G.6. For Eq. (B21), re-expressed here as Eq. (7), the equals condition is the maximum possible ε plus what we allow δ to be, since it comes directly from Eq. (B18). Eq. (B21) again is the result of developing the proposed bounding difference between the quantum and hidden-variable expectation values.

P.3. Now we see why we average variations limited by δ. With no variations, $A(\mathbf{b}, \lambda)\, B(\mathbf{b}, \lambda)$ is everywhere –1, and hence $|P(\mathbf{b}, \mathbf{b}) + 1|$ is everywhere zero. When we allow variation, $|\underline{P}(\mathbf{b}, \mathbf{b}) + 1|$ can be non-zero.

Next Bell presents a series of unnumbered equations. First, Bell uses Eq. (B19) to straightforwardly arrive at:

$$\underline{P}(\mathbf{a}, \mathbf{b}) - \underline{P}(\mathbf{a}, \mathbf{c}) = \int d\lambda\, \rho(\lambda)\, [\underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{b}, \lambda) - \underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{c}, \lambda)] \qquad (BX3)$$

In the above, the second expression on each side of the equation simply replaces **b** by **c** and includes a minus sign. Next, we rewrite Eq. (BX3) and see that Bell then manipulates Eq. (BX3) via:

$$\underline{P}(\mathbf{a}, \mathbf{b}) - \underline{P}(\mathbf{a}, \mathbf{c}) = \int d\lambda\, \rho(\lambda)\, \underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{b}, \lambda) - \int d\lambda\, \rho(\lambda)\, \underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{c}, \lambda)$$

$$= \int d\lambda\, \rho(\lambda)\, \underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{b}, \lambda)\, [1 + \underline{A}(\mathbf{b}, \lambda)\, \underline{B}(\mathbf{c}, \lambda)]$$

$$- \int d\lambda\, \rho(\lambda)\, \underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{c}, \lambda)\, [1 + \underline{A}(\mathbf{b}, \lambda)\, \underline{B}(\mathbf{b}, \lambda)] \qquad (BX4)$$

In Eq. (BX4) we can see that the portions corresponding to the 1 on the expressions to the right of the second equals sign correspond to what is on the left of the second equals sign, while the other two terms simply add and subtract $\int d\lambda\, \rho(\lambda)\, \underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{b}, \lambda)\, \underline{A}(\mathbf{b}, \lambda)\, \underline{B}(\mathbf{c}, \lambda)$ for no net effect.

Next Bell utilizes the fact that $|\underline{A}(\mathbf{a}, \lambda)|$, $|\underline{B}(\mathbf{b}, \lambda)|$ and $|\underline{B}(\mathbf{c}, \lambda)|$ are all <= 1 (they all represent averages over a range of measurements that can all only be +1 or –1) to obtain:

$$|\underline{P}(\mathbf{a}, \mathbf{b}) - \underline{P}(\mathbf{a}, \mathbf{c})| <= \int d\lambda\, \rho(\lambda)\, [1 + \underline{A}(\mathbf{b}, \lambda)\, \underline{B}(\mathbf{c}, \lambda)]$$

$$+ \int d\lambda\, \rho(\lambda)\, [1 + \underline{A}(\mathbf{b}, \lambda)\, \underline{B}(\mathbf{b}, \lambda)] \qquad \text{(BX5)}$$

There is again a typo (typo four) in that the original paper does not have $\rho(\lambda)$ in the first integral, as it has a proportional sign instead of ρ.

R.4. Typo four – the original paper does not have $\rho(\lambda)$ in the first integral of Eq. (BX5), as it has a proportional sign instead of ρ.

Eq. (BX5) follows from Eq. (BX4) and the conditions that $\underline{A}(\mathbf{a}, \lambda)$, $\underline{B}(\mathbf{b}, \lambda)$ and $\underline{B}(\mathbf{c}, \lambda)$ are all equal to plus or minus 1; and it is the change in sign over the integration range that leads these terms to be on average <1. The portions of the integrands inside the square brackets are always >= 0, so if we simply replace $\underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{b}, \lambda)$ by 1 and $\underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{c}, \lambda)$ by –1, those replacements into Eq. (BX4) will be non-negative and greater than or equal to what is replaced. This is what leads to Eq. (BX5).

G.7. An equality case for Eq. (BX5) is obtained when $\underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{b}, \lambda) = 1$ and $\underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{c}, \lambda) = -1$. This case can be obtained when **b** is antiparallel to **a** and **c** is parallel to **a**. But notice that in this case **b** – **c** is not small, instead **b** and **c** have a π difference in angle. Here we are no longer continuing to just develop the starting inequality; for the general case, needed later, equality will be obtained when we introduce a factor f <= 1 into Eq. (BX5):

$$|\underline{P}(\mathbf{a}, \mathbf{b}) - \underline{P}(\mathbf{a}, \mathbf{c})| = f\{\int d\lambda\, \rho(\lambda)\, [1 + \underline{A}(\mathbf{b}, \lambda)\, \underline{B}(\mathbf{c}, \lambda)]$$
$$+ \int d\lambda\, \rho(\lambda)\, [1 + \underline{A}(\mathbf{b}, \lambda)\, \underline{B}(\mathbf{b}, \lambda)]\} \qquad f<=1 \qquad (8)$$

In (8) f <= 1 is the factor that reduces from the maximum condition if we later choose values of **a**, **b** and **c** that are different from what it takes to achieve the maximum value obtained here. Note that f is zero when **b** = **c**.

R.5. Typo five – "using (19) and 21)" should be "using (19) and (21)" in the phrase before Eq. (BX6). This is truly minor, of course.

Bell then forms:

$$|\underline{P}(\mathbf{a}, \mathbf{b}) - \underline{P}(\mathbf{a}, \mathbf{c})| <= 1 + \underline{P}(\mathbf{b}, \mathbf{c}) + \varepsilon + \delta \qquad \text{(BX6)}$$

Eq. (BX6) comes from Eq. (BX5) by using $1 + \underline{P}(\mathbf{b}, \mathbf{c})$ from Eq. (B19), $\underline{P}(\mathbf{a}, \mathbf{b}) = \int d\lambda\, \rho(\lambda)\, \underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{b}, \lambda)$ (here Eq. (B19) becomes $\underline{P}(\mathbf{b}, \mathbf{c}) = \int d\lambda\, \rho(\lambda)\, \underline{A}(\mathbf{b}, \lambda)\, \underline{B}(\mathbf{c}, \lambda)$ and the first term on the right-hand side of Eq. (BX5) also provides a 1) and the "+ ε + δ" comes via Eq. (B21) (now our Eq. (7)) $\int d\lambda\, \rho(\lambda)\, [\underline{A}(\mathbf{b}, \lambda)\, \underline{B}(\mathbf{b}, \lambda) + 1] <= \varepsilon + \delta$.

G.8. The equality case for Eq. (BX6) follows from Eq. (8) just as Eq. (BX6) follows from Eq. (BX5):

$|\underline{P}(\mathbf{a}, \mathbf{b}) - \underline{P}(\mathbf{a}, \mathbf{c})| = f\{1 + \underline{P}(\mathbf{b}, \mathbf{c}) + \varepsilon + \delta\} \qquad f<=1 \qquad (9)$

P.4. If $\int d\lambda\, \rho(\lambda)\, [\underline{A}(\mathbf{b}, \lambda)\, \underline{B}(\mathbf{b}, \lambda) + 1]$ was zero, because we make a choice of $\delta = 0$, we'd just replace the $\varepsilon + \delta$ in Eqs. (BX6) and (9) by zero (see P.3.). This is relevant to P.10 below.

From Eq. (BX6) Bell then states that use of Eq. (B18), $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| <= \varepsilon + \delta$, will lead to his Eq. (BX7)

$|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - 2(\varepsilon + \delta) <= 1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta) \qquad \text{(BX7)}$

To get to Eq. (BX7) we begin by adding and subtracting $\mathbf{b} \cdot \mathbf{c}$ to Eq. (BX6)'s right-hand-side, while adding and subtracting $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \cdot \mathbf{c}$ to Eq. (BX6)'s left-hand-side:

$|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c}| <= 1 + \underline{P}(\mathbf{b}, \mathbf{c}) + \mathbf{b} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{c} + \varepsilon + \delta \qquad (10)$

The right-hand side of Eq. (10) can be made larger, and the left side smaller, and the inequality will still hold. If $\underline{P}(\mathbf{b}, \mathbf{c}) + \mathbf{b} \cdot \mathbf{c}$ is positive, then by Eq. (B18), $\underline{P}(\mathbf{b}, \mathbf{c}) + \mathbf{b} \cdot \mathbf{c} = |\underline{P}(\mathbf{b}, \mathbf{c}) + \mathbf{b} \cdot \mathbf{c}| <= \varepsilon + \delta$, and if it is negative it is even less, so we can replace $\underline{P}(\mathbf{b}, \mathbf{c}) + \mathbf{b} \cdot \mathbf{c}$ by $\varepsilon + \delta$ and the inequality still holds (since $\varepsilon + \delta$ is always $>= \underline{P}(\mathbf{b}, \mathbf{c}) + \mathbf{b} \cdot \mathbf{c}$), hence the right-hand side of Eq. (10) is allowed to become $1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta)$:

$|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c}| <= 1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta) \qquad (11)$

G.9. The equality condition of our replacement into Eq. (11) is found when $\underline{P}(\mathbf{b}, \mathbf{c}) + \mathbf{b} \cdot \mathbf{c}$ is positive and equal to $\varepsilon + \delta$. Notice here that the equality condition will involve **b** and **c**. We start with the equality given in G.8, $|\underline{P}(\mathbf{a}, \mathbf{b}) - \underline{P}(\mathbf{a}, \mathbf{c})| = f\{1 + \underline{P}(\mathbf{b}, \mathbf{c}) + \varepsilon + \delta\}$ and now we form

$|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c}| = f\{1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta)\} \qquad (12)$

P.5. Note that $\underline{P}(\mathbf{b}, \mathbf{c}) + \mathbf{b} \cdot \mathbf{c} = (\varepsilon + \delta)$ holds from: 1) our assumption that there is just such a bound (see discussion G.5); and 2) our taking of the positive case of $\underline{P}(\mathbf{b}, \mathbf{c}) + \mathbf{b} \cdot \mathbf{c}$. At this point in our development, the only possible contradiction in arriving at the equals condition because of possible conflicting choices of **a**, **b** and **c** is handled by our factor f. (Note that if we take the negative case of $\underline{P}(\mathbf{b}, \mathbf{c}) + \mathbf{b} \cdot \mathbf{c} = -(\varepsilon + \delta)$ then $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c}| = f\{1 - \mathbf{b} \cdot \mathbf{c}\}$. As seen below this would lead to a higher limit for $(\varepsilon + \delta)$ and we are seeking the lower limit.)

For the left-hand side of Eq. (11), we begin by introducing:

$\mathbf{d} = -\mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c} = -\mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) \qquad (13)$

With Eq. (13) we simplify the left-hand side of Eq. (11) to

$|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c}| = |\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}| \qquad (14)$

This leaves Eq. (11) as:

$|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}| <= 1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta)$ (15)

We'll now consider the possible cases of Eq. (15). Each case uses the relationship of Eq. (B18), $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| <= \varepsilon + \delta$. In the cases below, we look at what it takes for the left-hand side of Eq. (15) to be possibly even less but never greater, as the inequality will then still hold. We'll also identify the equality cases for further evaluation later.

P.6. Making the left-hand side of Eq. (15) less is relevant to G.18 below.

Case A1) if $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}$ is negative and $\mathbf{d}$ is negative, $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}$ will decrease the left-hand side of Eq. (15) from $\mathbf{d}$, leading to a larger absolute value than what we get from $\mathbf{d}$ alone, so $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}| >= |\mathbf{d}| - (\varepsilon + \delta)$ in this case. Here, equality is never obtained, only the greater than condition exists, and we can replace $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}|$ by $|\mathbf{d}| - (\varepsilon + \delta)$ since the latter is always less, and the inequality in Eq. (15) still holds.

P.7. Notice here that we could also replace $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}|$ by $|\mathbf{d}| - 2(\varepsilon + \delta)$, or by $|\mathbf{d}| - n(\varepsilon + \delta)$ where n is an arbitrarily large integer, and the inequality would still remain. But since we are trying to find the bound of ε it is important that an equals condition of the inequality exists somewhere within the domain of analysis. This equals condition will now be found in cases A2 and A3.

Useful in all cases is Eq. (B18), $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| <= \varepsilon + \delta$.

Case A2) $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}$ is negative and $\mathbf{d}$ is positive. $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}$ will decrease what is inside the absolute value from $\mathbf{d}$, with the decrease no larger than $\varepsilon + \delta$. There are sub-cases to consider here. Case A2.1) If $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| <= |\mathbf{d}|$ then $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}| >= |\mathbf{d}| - (\varepsilon + \delta)$ holds in this case. ($|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}| = |\mathbf{d}| - |\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}|$ in this case.) G.10. Equality is obtained when $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}| = |\mathbf{d}| - (\varepsilon + \delta)$; i.e., when $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} = -(\varepsilon + \delta)$. Case A2.2) If $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| >= |\mathbf{d}|$ then with $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| <= (\varepsilon + \delta)$, we have $|\mathbf{d}| <= |\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| <= (\varepsilon + \delta)$, so $|\mathbf{d}| <= (\varepsilon + \delta)$ and $|\mathbf{d}| - (\varepsilon + \delta) <= 0$, and hence $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}| >= |\mathbf{d}| - (\varepsilon + \delta)$ holds in this case since the left side is positive or zero and the right side is negative or zero. To search for equality, this can only happen when both sides are zero, or, G.11. Equality is obtained when $(\varepsilon + \delta) = |\mathbf{d}| = |\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}|$ and since case A.2 stipulates that $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}$ is negative we again get $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} = -(\varepsilon + \delta)$ for this case.

Case A3) if $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}$ is positive and $\mathbf{d}$ is negative, $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}$ will increase what is inside the absolute value, with the increase no larger than $\varepsilon + \delta$. There are sub-cases to consider here. Case A3.1) If $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| <= |\mathbf{d}|$ then $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}| >= |\mathbf{d}| - (\varepsilon + \delta)$ holds in this case. G.12. Equality is obtained when $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}| = |\mathbf{d}| - (\varepsilon + \delta)$; i.e., when $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} = (\varepsilon + \delta)$. Case A3.2) If $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| >= |\mathbf{d}|$, then with $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| <= (\varepsilon + \delta)$, we have $|\mathbf{d}| <= |\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| <= (\varepsilon + \delta)$, so $|\mathbf{d}| <= (\varepsilon + \delta)$ and $|\mathbf{d}| - (\varepsilon + \delta) <= 0$, and hence $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}| >= |\mathbf{d}| - (\varepsilon + \delta)$ holds in this case since the left side is positive or zero and the right side is negative or zero. To search for equality, this can only happen when both sides are zero, or, G.13. Equality is obtained when $(\varepsilon + \delta) = |\mathbf{d}| =$

$|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}|$ and since case A.3 stipulates that $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}$ is positive we again get $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} = (\varepsilon + \delta)$ for this case.

P.8. Similar to the discussion in section P.5, the relations $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} = -(\varepsilon + \delta)$ for Case A2 and $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} = (\varepsilon + \delta)$ for case A3 hold from our assumption that there is just such a bound (see discussion G.5). At this point in our development, the only possible contradiction in arriving at the equals condition because of possible conflicting choices of **a**, **b** and **c** is handled by our factor f. (See section G.7.) This is because we can specify one vector (for instance **c**) without loss of generality, and we can then specify **b** freely when we investigate $\underline{P}(\mathbf{b}, \mathbf{c}) + \mathbf{b} \cdot \mathbf{c}$ (see section P.5) and we can specify **a** freely when we investigate $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}$ here.

Case A4) if $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}$ is positive and **d** is positive, $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}$ will increase what is inside the absolute value, so $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}| >= |\mathbf{d}| - (\varepsilon + \delta)$ holds in this case. Here, equality is never obtained, only the greater than condition exists.

So, in call cases, $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}| >= |\mathbf{d}| - (\varepsilon + \delta)$ holds. And this allows us to replace $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}|$ by $|\mathbf{d}| - (\varepsilon + \delta)$ on the left-hand side of Eq. (15) since the latter is less than or equal and the inequality still holds. Repeating Eq. (15) and making the replacement:

$$|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} + \mathbf{d}| <= 1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta) \qquad (15)$$

$$|\mathbf{d}| - (\varepsilon + \delta) <= 1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta) \qquad (16)$$

Recalling Eq. (13) $\mathbf{d} = -\mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c})$, substituting **d** back in and adding and subtracting $\mathbf{a} \cdot \mathbf{c}$:

$$|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| - (\varepsilon + \delta) <= 1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta) \qquad (17)$$

Next, we must deal with $-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}$ on the left-hand side of Eq. (17) and we will again use Eq. (B18), $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| <= \varepsilon + \delta$ to examine the various cases. In the cases below, we look at what it takes for the left-hand side of Eq. (17) to be possibly even less, but never greater, as the inequality will then still hold. We'll also pull out the equality cases for further evaluation later.

P.9. Below we will find the relations $\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c} = g(\varepsilon + \delta)$ for Case B2 and $\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c} = -g(\varepsilon + \delta)$ for case B3. The factor g will come from our assertion that there is a bound of $(\varepsilon + \delta)$ (see discussion G.5) and because we may have a situation where the magnitude of the equals condition is less than that bound. This is because we have already specified **b** and **c** freely when we investigated $\underline{P}(\mathbf{b}, \mathbf{c}) + \mathbf{b} \cdot \mathbf{c}$ in comment P.5 and we specified **a** freely when we investigated $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}$ in comment P.8. But now, since **a**, **b** and **c** are already defined, it might be the case that only the less than portion of Eq. (B18) will exist here. Introducing a factor of $g <= 1$ accounts for this issue. Here, note that any asserted bound clearly is dependent on being able to vary **a**, **b** and **c** to find a maximum since at certain values we have seen that $\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c} = 0$. Now, the factor of g could have been introduced at earlier steps instead, either in our obtaining of Eq. (11), or in case A, but we choose to introduce it here in case B.

Case B1) Both $(-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c})$ and $(\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b})$ are negative. ($\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c}$ is positive.) In this case, $(-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c})$ will decrease the left-hand side of Eq. (17), and the absolute value is larger, so $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| >= |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - (\varepsilon + \delta)$. Here, equality is never obtained, only the greater than condition exists.

Case B2) $(-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c})$ is negative and $(\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b})$ is positive ($\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c}$ is positive). $(-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c})$ will decrease what is inside the absolute value, with the magnitude of the decrease no larger than $\varepsilon + \delta$. There are sub-cases to consider here. Case B2.1) If $|-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| <= |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}|$ then $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| >= |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - (\varepsilon + \delta)$ holds in this case. G.14. Equality is obtained when $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| = |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - g(\varepsilon + \delta)$; i.e., when $\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c} = g(\varepsilon + \delta)$. (The factor g is <= 1. See comment P.9 for more on g.) Case B2.2) If $|-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| >= |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}|$ then with $|-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| = \underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c} <= (\varepsilon + \delta)$, we have $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| <= |-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| <= (\varepsilon + \delta)$, so $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| <= (\varepsilon + \delta)$ and $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - (\varepsilon + \delta) <= 0$, and hence $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| >= |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - (\varepsilon + \delta)$ holds in this case since the left side is positive or zero and the right side is negative or zero. To search for equality, this can only happen when both sides are zero, or, G.15. Equality may be obtained when $(\varepsilon + \delta) = |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| = |\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c}|$ and since case B.2 stipulates that $\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c}$ is positive we may get $\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c} = (\varepsilon + \delta)$ for this case. However, we may get no equality in this case at all because we are not free to explore all possibilities for **a** and **c** independently of what we've already defined. (See comment P.9.)

Case B3) if $(-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c})$ is positive and $(\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b})$ is negative ($\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c}$ is negative). $(-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c})$ will increase what is inside the absolute value, with the increase no larger than $\varepsilon + \delta$. There are sub-cases to consider here. Case B3.1) If $|-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| <= |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}|$ then $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| >= |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - (\varepsilon + \delta)$ holds in this case. G.16. Equality is obtained when $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| = |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - g(\varepsilon + \delta)$; i.e., when $\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c} = -g(\varepsilon + \delta)$. (The factor g is <= 1. See comment P.9 for more on g.) Case B3.2) If $|-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| >= |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}|$ then with $|-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| = |\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c}| <= (\varepsilon + \delta)$, we have $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| <= |-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| <= (\varepsilon + \delta)$, so $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| <= (\varepsilon + \delta)$ and $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - (\varepsilon + \delta) <= 0$, and hence $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| >= |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - (\varepsilon + \delta)$ holds in this case since the left side is positive or zero and the right side is negative or zero. To search for equality, this can only happen when both sides are zero, or, G.17. Equality may be obtained when $(\varepsilon + \delta) = |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| = |\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c}|$ and since case B.3 stipulates that $\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c}$ is negative we may get $\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c} = -(\varepsilon + \delta)$ for this case. However, we may get no equality in this case at all because we are not free to explore all possibilities for **a** and **c** independently of what we've already defined. (See comment P.9.)

Case B4) if ($-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}$) is positive and ($\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}$) is positive ($\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c}$ is negative). ($-\underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}$) will increase what is inside the absolute value, so $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| >= |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - (\varepsilon + \delta)$ holds in this case. Here, equality is never obtained, only the greater than condition exists.

So, in call cases, $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}| >= |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - (\varepsilon + \delta)$ holds. And this allows us to replace $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c}|$ by $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - (\varepsilon + \delta)$ on the left-hand side of Eq. (17):

$$|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - 2(\varepsilon + \delta) <= 1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta) \qquad \text{(BX7)}$$

Indeed, Bell's Eq. (BX7) is correct, but there is a lot to consider in proving this to be the case.

G.18. Equality analysis. To find equalities within Eq. (BX7) we start with Eq. (12):

$$|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} - \underline{P}(\mathbf{a}, \mathbf{c}) - \mathbf{a} \cdot \mathbf{c} + \mathbf{a} \cdot \mathbf{c}| = f\{1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta)\} \qquad (12)$$

We have four cases to evaluate.

C1. First, we'll analyze the situation using cases A3 and B3. From A3 we see equality is obtained when $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} = (\varepsilon + \delta)$ and from B3 we see equality is obtained when $\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c} = -g(\varepsilon + \delta)$. Hence Eq. (12) becomes

$$|(\varepsilon + \delta) - \mathbf{a} \cdot \mathbf{b} + g(\varepsilon + \delta) + \mathbf{a} \cdot \mathbf{c}| = f\{1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta)\} \qquad (18)$$

$\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}$ is negative, as specified in case B3. $(\varepsilon + \delta)$ is positive, so we get two subcases.

Subcase C1.1. For this subcase $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| >= (\varepsilon + \delta) + g(\varepsilon + \delta)$ so the equality condition is:

$$|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - (\varepsilon + \delta) - g(\varepsilon + \delta) = f\{1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta)\} \qquad (19)$$

We get:

$$(\varepsilon + \delta) + g(\varepsilon + \delta) + 2f(\varepsilon + \delta) = |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - f(1 - \mathbf{b} \cdot \mathbf{c}) \qquad (20)$$

This becomes:

$$(\varepsilon + \delta) = [|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - f(1 - \mathbf{b} \cdot \mathbf{c})] / (1 + g + 2f) \qquad (21)$$

Subcase C1.2. For this subcase $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| < (\varepsilon + \delta) + g(\varepsilon + \delta)$ so the attempted equality condition of Eq. (18) becomes:

$$(\varepsilon + \delta) + g(\varepsilon + \delta) - |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| = f\{1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta)\} \qquad (22)$$

We get:

$$(\varepsilon + \delta) + g(\varepsilon + \delta) - 2f(\varepsilon + \delta) = |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| + f(1 - \mathbf{b} \cdot \mathbf{c}) \qquad (23)$$

$$(\varepsilon + \delta) = [|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| + f(1 - \mathbf{b} \cdot \mathbf{c})] / (1 + g - 2f) \qquad (24)$$

Eq. (21) has a smaller or equal numerator and larger or equal denominator than Eq. (24) indicating that the value found in subcase C1.1 is smaller than that found here in section C1.2. Since we are seeking the smallest value, we need not evaluate subcase C1.2 any further.

C2. Next, we'll analyze the situation using cases A2 and B3. From A2 we see equality is obtained when $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} = -(\varepsilon + \delta)$ and from B3 we see equality is obtained when $\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c} = -g(\varepsilon + \delta)$. Eq. (12) becomes:

$$|-(\varepsilon + \delta) - \mathbf{a} \cdot \mathbf{b} + g(\varepsilon + \delta) + \mathbf{a} \cdot \mathbf{c}| = f\{1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta)\} \quad (25)$$

We know $\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}$ must be negative, as it is required by B3. Also $-(\varepsilon + \delta) + g(\varepsilon + \delta)$ is non-positive since $g <= 1$. Hence, Eq. (25) becomes that which has its signs reversed within the absolute value on the left-hand side:

$$(\varepsilon + \delta) + \mathbf{a} \cdot \mathbf{b} - g(\varepsilon + \delta) - \mathbf{a} \cdot \mathbf{c} = f\{1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta)\} \quad (26)$$

$$2f(\varepsilon + \delta) + (g-1)(\varepsilon + \delta) = |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - f(1 - \mathbf{b} \cdot \mathbf{c}) \quad (27)$$

$$(\varepsilon + \delta) = [|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - f(1 - \mathbf{b} \cdot \mathbf{c})] / (g - 1 + 2f) \quad (28)$$

Eq. (21) has the same numerator and a larger or equal denominator than Eq. (28) indicating that the value found in subcase C1.1 is smaller than that found here in section C2. Since we are seeking the smallest value, we need not evaluate subcase C2 any further.

C3. Next, we'll analyze the situation using cases A3 and B2. From A3 we see equality is obtained when $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} = (\varepsilon + \delta)$ and from B2 we see equality is obtained when $\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c} = g(\varepsilon + \delta)$. Hence Eq. (12) becomes:

$$|(\varepsilon + \delta) - \mathbf{a} \cdot \mathbf{b} - g(\varepsilon + \delta) + \mathbf{a} \cdot \mathbf{c}| = f\{1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta)\} \quad (29)$$

We know $\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}$ is positive as it is required by B2, and $(\varepsilon + \delta) - g(\varepsilon + \delta)$ is non-negative also so Eq. (29) becomes:

$$2f(\varepsilon + \delta) + (g-1)(\varepsilon + \delta) = |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - f(1 - \mathbf{b} \cdot \mathbf{c}) \quad (30)$$

$$(\varepsilon + \delta) = [|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - f(1 - \mathbf{b} \cdot \mathbf{c})] / (g - 1 + 2f) \quad (31)$$

Eq. (21) has the same numerator and a larger or equal denominator than Eq. (31) indicating that the value found in subcase C1.1 is smaller than that found here in section C3. Since we are seeking the smallest value, we need not evaluate subcase C3 any further.

C4. Finally, we'll analyze the situation using cases A2 and B2. From A2 we see equality is obtained when $\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b} = -(\varepsilon + \delta)$ and from B2 we see equality is obtained when $\underline{P}(\mathbf{a}, \mathbf{c}) + \mathbf{a} \cdot \mathbf{c} = g(\varepsilon + \delta)$. Hence Eq. (12) becomes

$$|-(\varepsilon + \delta) - \mathbf{a} \cdot \mathbf{b} - g(\varepsilon + \delta) + \mathbf{a} \cdot \mathbf{c}| = f\{1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta)\} \tag{32}$$

C4.1. Assuming $\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}$ is positive (as required by case B2) and $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| >= (1 + g)(\varepsilon + \delta)$ the equality condition is:

$$|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - (1 + g)(\varepsilon + \delta) = f\{1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta)\} \tag{33}$$

So, we get:

$$(1 + g)(\varepsilon + \delta) + 2f(\varepsilon + \delta) = |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - f(1 - \mathbf{b} \cdot \mathbf{c}) \tag{34}$$

$$(\varepsilon + \delta) = [|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - f(1 - \mathbf{b} \cdot \mathbf{c})] / (1 + g + 2f) \tag{35}$$

Notice that Eq. (35) is identical to Eq. (21), adding nothing new.

C4.2. Assuming $\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}$ is positive (as required by case B2) and $|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| < (1 + g)(\varepsilon + \delta)$ the equality condition is:

$$(1 + g)(\varepsilon + \delta) - |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| = f\{1 - \mathbf{b} \cdot \mathbf{c} + 2(\varepsilon + \delta)\} \tag{36}$$

So, we get:

$$(1 + g)(\varepsilon + \delta) - 2f(\varepsilon + \delta) = |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| + f(1 - \mathbf{b} \cdot \mathbf{c}) \tag{37}$$

$$(\varepsilon + \delta) = [|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| + f(1 - \mathbf{b} \cdot \mathbf{c})] / (1 + g - 2f) \tag{38}$$

Notice that Eq. (38) is identical to Eq. (24), adding nothing new.

Let's now recap the various case analyses. Any case of $\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}$ can be realized, because we can vary **a**, **b** and **c** over their ranges. Then recall Eq. (B18) $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| <= \varepsilon + \delta$, and Eq. (B19): $\underline{P}(\mathbf{a}, \mathbf{b}) = \int d\lambda\, \rho(\lambda)\, \underline{A}(\mathbf{a}, \lambda)\, \underline{B}(\mathbf{b}, \lambda)$. If we make $(\varepsilon + \delta)$ large enough we can achieve all of our possible-cases A and B above. Of those cases, our goal is to find the minimum $(\varepsilon + \delta)$ that can possibly be found. It is possible that the $\rho(\lambda)$, $\underline{A}(\mathbf{a}, \lambda)$ and $\underline{B}(\mathbf{b}, \lambda)$ found in that minimum condition are the ones found by nature. While there may be greater $(\varepsilon + \delta)$ values found in other cases, we needn't concern ourselves with those possibilities since the important point is to find the smallest possible one, make an evaluation of its equals condition, and show that it can't be arbitrarily small, proving the contradiction between the predictions of quantum mechanics and that of any hidden variable possibility. Also relevant is that $(\varepsilon + \delta)$ is a limit given through Eq. (B18), $|\underline{P}(\mathbf{a}, \mathbf{b}) + \mathbf{a} \cdot \mathbf{b}| <= \varepsilon + \delta$, which is bounded by 0 and 2. Hence only solutions $0 < (\varepsilon + \delta) < 2$ are of concern, and for that range we see that the equality conditions in cases C1.2, C.2, C.3 and C.4.2 all lead to a larger or equal value of $(\varepsilon + \delta)$ than the equality condition in case C.1.1. Hence, case C.1.1 is the minimum we seek, and it is not arbitrarily small.

P.10. The equality analysis of G.18 shows that several equality cases are theoretically possible that exceed Bell's limit of Eq. (BX7). This of course still leaves Bell's result of Eq. (BX7) valid. The many equality cases result from different conditions on **a**, **b**, **c**, $\rho(\lambda)$, $\underline{A}(\mathbf{b}, \lambda)$, and $\underline{B}(\mathbf{b}, \lambda)$ and our goal (following Bell) was to show that there is some minimum value of $(\varepsilon + \delta)$ that can't be made arbitrarily small, which we found as the equality condition given in Eq. (21), $(\varepsilon + \delta) = [|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - f(1 - \mathbf{b} \cdot \mathbf{c})] / (1 + g + 2f)$. At this point we can compare our Eq. (21) with Bell's Eq. (B22):

$$4(\varepsilon + \delta) >= |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| + \mathbf{b} \cdot \mathbf{c} - 1 \qquad \text{(B22)}$$

It can be seen that for f = g = 1, the limiting condition obtained by our Eq. (21) is identical to the condition obtained by Bell.

Next, recall P.4. where we noted that if $\int d\lambda\, \rho(\lambda)\, [\underline{A}(\mathbf{b}, \lambda)\, \underline{B}(\mathbf{b}, \lambda) + 1]$ was zero (because we chose $\delta = 0$) we'd just replace the $\varepsilon + \delta$ in Eqs. (BX6) and (9) by zero. This would eliminate one factor of $(\varepsilon + \delta)$ on the right-hand side of Eq. (BX7), as well as eliminating a factor of $(\varepsilon + \delta)$ from the right-hand sides of Eqs. (10), (11), (12), (15), (16), (17), (18), (19), (22), (25), (26), (29), (32), (33), and (36) and this would, in turn, increase the limiting value of ε in the equality analysis: it reduces 2f to f in all the equality statements. In that case our limiting condition for f = g = 1 becomes

$$3(\varepsilon + \delta) >= |\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| + \mathbf{b} \cdot \mathbf{c} - 1 \qquad \text{(39)}$$

which leads to a larger discrepancy between a hidden variable theory and quantum mechanics than that obtained by Bell. Furthermore, if f and g are less than one, the discrepancy becomes even larger. Since f and g would only make the discrepancy greater, we'll not look further into those factors here, being satisfied that not only is Bell correct that there is a discrepancy, but that the discrepancy is even stronger than the one found in his famous paper.

It is also of some interest to consider Bell's final unnumbered equation:

$$4(\varepsilon + \delta) >= \text{sqrt}(2) - 1 \qquad \text{(BX8)}$$

Bell obtains Eq. (BX8) through an example where he sets $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \text{sqrt}(2)/2$. Here we've found Eq. (21), $(\varepsilon + \delta) = [|\mathbf{a} \cdot \mathbf{c} - \mathbf{a} \cdot \mathbf{b}| - f(1 - \mathbf{b} \cdot \mathbf{c})] / (1 + g + 2f)$, and using the same examples for the dot products we get $(1 + g + 2f)(\varepsilon + \delta) = \text{sqrt}(2)/2 + f\, \text{sqrt}(2)/2 - f$. For f = g = 1 we obtain the same result as Bell for the equals condition. For f and g less than 1 we obtain a larger value for $(\varepsilon + \delta)$, and using Eq. (39) leaves an ever larger value for $(\varepsilon + \delta)$.

As a final comparison, let's recall the discrepancy between the hidden variable theory and the quantum mechanics prediction that we found at the end of section 3: $D = \cos\theta - (1 - 2\theta/\pi) = \text{sqrt}(2)/2 - 1/2$. We see that this value is twice the value Bell achieves in Eq. (BX8). It is also larger than what we obtain through Eq. (39).

8. A Review of the Typos.

Here are the typos found in this analysis:

R1. Typo One – Bell's erroneous second of Eqs. (B9), which should instead be our Eq. (3).

R.2. Typo Two – Bell's Eq. (B16) should be replaced by our Eq. (6).

R.3. Typo Three – Eq. (B21) should include an integral sign like we do in Eq. (7).

R.4. Typo four – the original paper does not have $\rho(\lambda)$ in the first integral of Eq. (BX5), as it has a proportional sign instead of ρ.

R.5. Typo five – "using (19) and 21)" should be "using (19) and (21)" in the phrase before Eq. (BX6). This is truly minor, of course.

The middle three of these typos are known. After I had found them, I also found this reference from CERN: https://cds.cern.ch/record/111654/files/vol1p195-200_001.pdf which corrects the middle three typos in the same way as done herein. The first typo is also clearly just a typo, and the fifth is an exceedingly minor presentation issue.

9. Conclusion.

In conclusion, we find that Bell's 1964 paper arrives at the correct conclusion that hidden variable expectation values are in contradiction with quantum mechanical expectation values for the case of entangled states. This of course should come as no surprise, since it has been a topic of intense interest since the 1964 paper was written. The 1964 paper does have several typos however, and it is furthermore also quite terse, which may make it difficult for students to fully derive its correctness. In this work, I've pointed out corrections to the typos, added many additional intermediate steps in the development of the equations, as well as specified the important equals conditions within the famous inequalities. I hope this work may therefore make it easier for students to understand Bell's important 1964 paper. We also found an even greater discrepancy between hidden variable and quantum mechanical expectation values than that given by Bell once we eliminate the δ variation that he made in his analysis.

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ON THE EINSTEIN PODOLSKY ROSEN PARADOX*

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I. Introduction

THE paradox of Einstein, Podolsky and Rosen [1] was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional variables. These additional variables were to restore to the theory causality and locality [2]. In this note that idea will be formulated mathematically and shown to be incompatible with the statistical predictions of quantum mechanics. It is the requirement of locality, or more precisely that the result of a measurement on one system be unaffected by operations on a distant system with which it has interacted in the past, that creates the essential difficulty. There have been attempts [3] to show that even without such a separability or locality requirement no "hidden variable" interpretation of quantum mechanics is possible. These attempts have been examined elsewhere [4] and found wanting. Moreover, a hidden variable interpretation of elementary quantum theory [5] has been explicitly constructed. That particular interpretation has indeed a grossly non-local structure. This is characteristic, according to the result to be proved here, of any such theory which reproduces exactly the quantum mechanical predictions.

II. Formulation

With the example advocated by Bohm and Aharonov [6], the EPR argument is the following. Consider a pair of spin one-half particles formed somehow in the singlet spin state and moving freely in opposite directions. Measurements can be made, say by Stern-Gerlach magnets, on selected components of the spins $\vec{\sigma}_1$ and $\vec{\sigma}_2$. If measurement of the component $\vec{\sigma}_1 \cdot \vec{a}$, where $\vec{a}$ is some unit vector, yields the value $+1$ then, according to quantum mechanics, measurement of $\vec{\sigma}_2 \cdot \vec{a}$ must yield the value -1 and vice versa. Now we make the hypothesis [2], and it seems one at least worth considering, that if the two measurements are made at places remote from one another the orientation of one magnet does not influence the result obtained with the other. Since we can predict in advance the result of measuring any chosen component of $\vec{\sigma}_2$, by previously measuring the same component of $\vec{\sigma}_1$, it follows that the result of any such measurement must actually be predetermined. Since the initial quantum mechanical wave function does *not* determine the result of an individual measurement, this predetermination implies the possibility of a more complete specification of the state.

Let this more complete specification be effected by means of parameters λ. It is a matter of indifference in the following whether λ denotes a single variable or a set, or even a set of functions, and whether the variables are discrete or continuous. However, we write as if λ were a single continuous parameter. The result A of measuring $\vec{\sigma}_1 \cdot \vec{a}$ is then determined by $\vec{a}$ and λ, and the result B of measuring $\vec{\sigma}_2 \cdot \vec{b}$ in the same instance is determined by $\vec{b}$ and λ, and

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$$A(\vec{a}, \lambda) = \pm 1, \; B(\vec{b}, \lambda) = \pm 1. \tag{1}$$

The vital assumption [2] is that the result B for particle 2 does not depend on the setting $\vec{a}$, of the magnet for particle 1, nor A on $\vec{b}$.

If $\rho(\lambda)$ is the probability distribution of λ then the expectation value of the product of the two components $\vec{\sigma}_1 \cdot \vec{a}$ and $\vec{\sigma}_2 \cdot \vec{b}$ is

$$P(\vec{a}, \vec{b}) = \int d\lambda \, \rho(\lambda) A(\vec{a}, \lambda) B(\vec{b}, \lambda) \tag{2}$$

This should equal the quantum mechanical expectation value, which for the singlet state is

$$\langle \vec{\sigma}_1 \cdot \vec{a} \; \vec{\sigma}_2 \cdot \vec{b} \rangle = -\vec{a} \cdot \vec{b} \, . \tag{3}$$

But it will be shown that this is not possible.

Some might prefer a formulation in which the hidden variables fall into two sets, with A dependent on one and B on the other; this possibility is contained in the above, since λ stands for any number of variables and the dependences thereon of A and B are unrestricted. In a complete physical theory of the type envisaged by Einstein, the hidden variables would have dynamical significance and laws of motion; our λ can then be thought of as initial values of these variables at some suitable instant.

III. Illustration

The proof of the main result is quite simple. Before giving it, however, a number of illustrations may serve to put it in perspective.

Firstly, there is no difficulty in giving a hidden variable account of spin measurements on a single particle. Suppose we have a spin half particle in a pure spin state with polarization denoted by a unit vector $\vec{p}$. Let the hidden variable be (for example) a unit vector $\vec{\lambda}$ with uniform probability distribution over the hemisphere $\vec{\lambda} \cdot \vec{p} > 0$. Specify that the result of measurement of a component $\vec{\sigma} \cdot \vec{a}$ is

$$\text{sign } \vec{\lambda} \cdot \vec{a}' \, , \tag{4}$$

where $\vec{a}'$ is a unit vector depending on $\vec{a}$ and $\vec{p}$ in a way to be specified, and the sign function is $+1$ or -1 according to the sign of its argument. Actually this leaves the result undetermined when $\lambda \cdot a' = 0$, but as the probability of this is zero we will not make special prescriptions for it. Averaging over $\vec{\lambda}$ the expectation value is

$$\langle \vec{\sigma} \cdot \vec{a} \rangle = 1 - 2\theta'/\pi \, , \tag{5}$$

where θ' is the angle between $\vec{a}'$ and $\vec{p}$. Suppose then that $\vec{a}'$ is obtained from $\vec{a}$ by rotation towards $\vec{p}$ until

$$1 - \frac{2\theta'}{\pi} = \cos\theta \tag{6}$$

where θ is the angle between $\vec{a}$ and $\vec{p}$. Then we have the desired result

$$\langle \vec{\sigma} \cdot \vec{a} \rangle = \cos\theta \tag{7}$$

So in this simple case there is no difficulty in the view that the result of every measurement is determined by the value of an extra variable, and that the statistical features of quantum mechanics arise because the value of this variable is unknown in individual instances.

Secondly, there is no difficulty in reproducing, in the form (2), the only features of (3) commonly used in verbal discussions of this problem:

$$\left.\begin{aligned} P(\vec{a}, \vec{a}) &= -P(\vec{a}, -\vec{a}) = -1 \\ P(\vec{a}, \vec{b}) &= 0 \text{ if } \vec{a}\cdot\vec{b} = 0 \end{aligned}\right\} \tag{8}$$

For example, let λ now be unit vector $\vec{\lambda}$, with uniform probability distribution over all directions, and take

$$\left.\begin{aligned} A(\vec{a}, \vec{\lambda}) &= \text{sign } \vec{a}\cdot\vec{\lambda} \\ B(a, b) &= -\text{sign } \vec{b}\cdot\vec{\lambda} \end{aligned}\right\} \tag{9}$$

This gives

$$P(\vec{a}, \vec{b}) = -1 + \frac{2}{\pi}\theta\,, \tag{10}$$

where θ is the angle between a and b, and (10) has the properties (8). For comparison, consider the result of a modified theory [6] in which the pure singlet state is replaced in the course of time by an isotropic mixture of product states; this gives the correlation function

$$-\frac{1}{3}\vec{a}\cdot\vec{b} \tag{11}$$

It is probably less easy, experimentally, to distinguish (10) from (3), than (11) from (3).

Unlike (3), the function (10) is not stationary at the minimum value -1 (at $\theta = 0$). It will be seen that this is characteristic of functions of type (2).

Thirdly, and finally, there is no difficulty in reproducing the quantum mechanical correlation (3) if the results A and B in (2) are allowed to depend on $\vec{b}$ and $\vec{a}$ respectively as well as on $\vec{a}$ and $\vec{b}$. For example, replace $\vec{a}$ in (9) by $\vec{a}'$, obtained from $\vec{a}$ by rotation towards $\vec{b}$ until

$$1 - \frac{2}{\pi}\theta' = \cos\theta\,,$$

where θ' is the angle between $\vec{a}'$ and $\vec{b}$. However, for given values of the hidden variables, the results of measurements with one magnet now depend on the setting of the distant magnet, which is just what we would wish to avoid.

IV. Contradiction

The main result will now be proved. Because ρ is a normalized probability distribution,

$$\int d\lambda\,\rho(\lambda) = 1, \tag{12}$$

and because of the properties (1), P in (2) cannot be less than -1. It can reach -1 at $\vec{a} = \vec{b}$ only if

$$A(\vec{a}, \lambda) = -B(\vec{a}, \lambda) \tag{13}$$

except at a set of points λ of zero probability. Assuming this, (2) can be rewritten

$$P(\vec{a}, \vec{b}) = -\int d\lambda\,\rho(\lambda)\,A(\vec{a}, \lambda)\,A(\vec{b}, \lambda). \tag{14}$$

It follows that $\vec{c}$ is another unit vector

$$P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c}) = -\int d\lambda\, \rho(\lambda)\, [A(\vec{a}, \lambda)\, A(\vec{b}, \lambda) - A(\vec{a}, \lambda)\, A(\vec{c}, \lambda)]$$
$$= \int d\lambda\, \rho(\lambda)\, A(\vec{a}, \lambda)\, A(\vec{b}, \lambda)\, [A(\vec{b}, \lambda)\, A(\vec{c}, \lambda) - 1]$$

using (1), whence

$$|P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c})| \leq \int d\lambda\, \rho(\lambda)\, [1 - A(\vec{b}, \lambda)\, A(\vec{c}, \lambda)]$$

The second term on the right is $P(\vec{b}, \vec{c})$, whence

$$1 + P(\vec{b}, \vec{c}) \geq |P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{c})| \tag{15}$$

Unless P is constant, the right hand side is in general of order $|\vec{b} - \vec{c}|$ for small $|\vec{b} - \vec{c}|$. Thus $P(\vec{b}, \vec{c})$ cannot be stationary at the minimum value (-1 at $\vec{b} = \vec{c}$) and cannot equal the quantum mechanical value (3).

Nor can the quantum mechanical correlation (3) be arbitrarily closely approximated by the form (2). The formal proof of this may be set out as follows. We would not worry about failure of the approximation at isolated points, so let us consider instead of (2) and (3) the functions

$$\bar{P}(\vec{a}, \vec{b}) \text{ and } \overline{-\vec{a} \cdot \vec{b}}$$

where the bar denotes independent averaging of $P(\vec{a}', \vec{b}')$ and $-\vec{a}' \cdot \vec{b}'$ over vectors $\vec{a}'$ and $\vec{b}'$ within specified small angles of $\vec{a}$ and $\vec{b}$. Suppose that for all $\vec{a}$ and $\vec{b}$ the difference is bounded by ϵ:

$$|\bar{P}(\vec{a}, \vec{b}) + \vec{a} \cdot \vec{b}| \leq \epsilon \tag{16}$$

Then it will be shown that ϵ cannot be made arbitrarily small.

Suppose that for all a and b

$$|\overline{\vec{a} \cdot \vec{b}} - \vec{a} \cdot \vec{b}| \leq \delta \tag{17}$$

Then from (16)

$$|\bar{P}(\vec{a}, \vec{b}) + \vec{a} \cdot \vec{b}| \leq \epsilon + \delta \tag{18}$$

From (2)

$$\bar{P}(\vec{a}, \vec{b}) = \int d\lambda\, \rho(\lambda)\, \bar{A}(\vec{a}, \lambda)\, \bar{B}(\vec{b}, \lambda) \tag{19}$$

where

$$|\bar{A}(\vec{a}, \lambda)| \leq 1 \text{ and } |\bar{B}(\vec{b}, \lambda)| \leq 1 \tag{20}$$

From (18) and (19), with $\vec{a} = \vec{b}$,

$$d\lambda\, \rho(\lambda)\, [\bar{A}(\vec{b}, \lambda)\, \bar{B}(\vec{b}, \lambda) + 1] \leq \epsilon + \delta \tag{21}$$

From (19)

$$\bar{P}(\vec{a}, \vec{b}) - \bar{P}(\vec{a}, \vec{c}) = \int d\lambda\, \rho(\lambda)\, [\bar{A}(\vec{a}, \lambda)\, \bar{B}(\vec{b}, \lambda) - \bar{A}(\vec{a}, \lambda)\, \bar{B}(\vec{c}, \lambda)]$$
$$= \int d\lambda\, \rho(\lambda)\, \bar{A}(\vec{a}, \lambda)\, \bar{B}(\vec{b}, \lambda)\, [1 + \bar{A}(\vec{b}, \lambda)\, \bar{B}(\vec{c}, \lambda)]$$
$$- \int d\lambda\, \rho(\lambda)\, \bar{A}(\vec{a}, \lambda)\, \bar{B}(\vec{c}, \lambda)\, [1 + \bar{A}(\vec{b}, \lambda)\, \bar{B}(\vec{b}, \lambda)]$$

Using (20) then

$$|\bar{P}(\vec{a}, \vec{b}) - \bar{P}(\vec{a}, \vec{c})| \leq \int d\lambda \propto(\lambda) [1 + \bar{A}(\vec{b}, \lambda) \bar{B}(\vec{c}, \lambda)] + \int d\lambda \rho(\lambda) [1 + \bar{A}(\vec{b}, \lambda) \bar{B}(\vec{b}, \lambda)]$$

Then using (19) and 21)

$$|\bar{P}(\vec{a}, \vec{b}) - \bar{P}(\vec{a}, \vec{c})| \leq 1 + \bar{P}(\vec{b}, \vec{c}) + \epsilon + \delta$$

Finally, using (18),

$$|\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b}| - 2(\epsilon + \delta) \leq 1 - \vec{b} \cdot \vec{c} + 2(\epsilon + \delta)$$

or

$$4(\epsilon + \delta) \geq |\vec{a} \cdot \vec{c} - \vec{a} \cdot \vec{b}| + \vec{b} \cdot \vec{c} - 1 \tag{22}$$

Take for example $\vec{a} \cdot \vec{c} = 0$, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = 1/\sqrt{2}$ Then

$$4(\epsilon + \delta) \geq \sqrt{2} - 1$$

Therefore, for small finite δ, ϵ cannot be arbitrarily small.

Thus, the quantum mechanical expectation value cannot be represented, either accurately or arbitrarily closely, in the form (2).

V. Generalization

The example considered above has the advantage that it requires little imagination to envisage the measurements involved actually being made. In a more formal way, assuming [7] that any Hermitian operator with a complete set of eigenstates is an "observable", the result is easily extended to other systems. If the two systems have state spaces of dimensionality greater than 2 we can always consider two dimensional subspaces and define, in their direct product, operators $\vec{\sigma}_1$ and $\vec{\sigma}_2$ formally analogous to those used above and which are zero for states outside the product subspace. Then for at least one quantum mechanical state, the "singlet" state in the combined subspaces, the statistical predictions of quantum mechanics are incompatible with separable predetermination.

VI. Conclusion

In a theory in which parameters are added to quantum mechanics to determine the results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, however remote. Moreover, the signal involved must propagate instantaneously, so that such a theory could not be Lorentz invariant.

Of course, the situation is different if the quantum mechanical predictions are of limited validity. Conceivably they might apply only to experiments in which the settings of the instruments are made sufficiently in advance to allow them to reach some mutual rapport by exchange of signals with velocity less than or equal to that of light. In that connection, experiments of the type proposed by Bohm and Aharonov [6], in which the settings are changed during the flight of the particles, are crucial.

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